Dissipative phase-fluctuations in superconducting wires capacitively coupled to diffusive metals

Alejandro M. Lobos and Thierry Giamarchi
DPMC-MaNEP, University of Geneva, 24 Quai Ernest-Ansermet CH-1211 Geneva, Switzerland.
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We study the screening of the Coulomb interaction in a quasi one-dimensional superconductor given by the presence of either a one- or a two-dimensional non-interacting electron gas. To that end, we derive an effective low-energy phase-only action, which amounts to treating the Coulomb and superconducting correlations in the random-phase approximation. We concentrate on the study of dissipation effects in the superconductor, induced by the effect of Coulomb coupling to the diffusive density-modes in the metal, and study its consequences on the static and dynamic conductivity. Our results point towards the importance of the dimensionality of the screening metal in the behavior of the superconducting plasma mode of the wire at low energies. In absence of topological defects, and when the screening is given by a one-dimensional electron gas, the superconducting plasma mode is completely damped in the limit $q \to 0$, and consequently superconductivity is lost in the wire. In contrast, we recover a Drude-response in the conductivity when the screening is provided by a two-dimensional electron gas.

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I. INTRODUCTION

The environment has profound effects on the properties of quantum systems[1]. In the case of superconductors, it was predicted more than 25 years ago that a resistively shunted Josephson junction would experience a superconductor-normal transition as a function of R_S/R_Q, where R_S is the shunt resistance of the junction and $R_Q = h/4e^2 \approx 6.45$ kΩ is the quantum of resistance[2–4]. More recently, a variety of superconducting systems, including granular[5] , or homogeneous[6] films, 2D Josephson junctions arrays[7], out-of-equilibrium Josephson junctions[8] and high temperature superconductors[9] were shown to undergo a superconductor-insulator transition as the characteristic resistance of the system in the normal state increases through a critical value on the order of R_Q. In those cases, the dissipative environment corresponds to the measurement circuits or the intrinsic component of normal electrons in the system.

In contrast, isolated superconducting wires with lateral dimension $r_0 \ll \xi_0$, where ξ_0 is the bulk coherence length, do not present significant dissipation sources at low temperatures. The low-energy modes in an ideally isolated superconducting wire are the one-dimensional propagating plasmon modes along the axis[10]. Contrary to bulk superconductors, where the plasmon has an energy $\omega_p^{3\mathrm{D}} = \sqrt{4\pi n_s e^2/m}$ (where n_s is the superfluid density and m is the electron mass), in the restricted 1D geometry of the wire, the long-range Coulomb interaction is not completely screened and consequently charge fluctuations are not shifted to finite energies in the limit $q \to 0$. The result is a sound-like dispersion relation $\omega^2(q) \sim q^2 \ln(1/qr_0)$, where the logarithmic factor is a remnant of the long-range Coulomb interactions.

Because of the gapless dispersion relation, quantum fluctuations are expected to show critical behavior[11], a feature that has attracted the attention of several theoretical[12–15] and experimental[16–18] research groups.

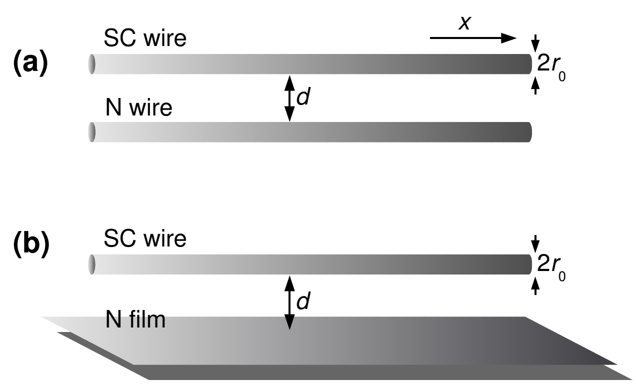

Figure 1: Representation of the capacitively coupled superconducting wire-normal metal system. The metal placed at a distance d screens the long-range Coulomb interaction in the superconducting wire. In (a) the metal is a diffusive wire, and in (b) we consider a diffusive 2D electron gas .

How this picture (i.e., sound-like dispersion relation and critical behavior) is modified when the coupling to the electromagnetic environment is taken into account? Intuitively, the presence of a metal at a distance d should screen the long-range Coulomb interaction for density fluctuations with wavelength $q \ll d^{-1}$, resulting in enhanced superconducting correlations[11]. On the other hand, in capacitively-coupled superconductor-normal systems, the presence of dissipation in the normal metal is known to produce dissipative order-parameter fluctuations[19–21] and, from this point of view, screening might also carry detrimental effects to superconductivity. Moreover, recent theoretical works on related Luttinger-

liquid systems coupled electrostatically to metals predict charge-density wave (CDW) instabilities caused by the dissipative environment[22,23].

Therefore, a better understanding of the screening effects occurring in superconducting wires and the consequences to their superconducting properties is needed. This issue is particularly relevant to recent theoretical[13,24,25] and experimental[26,27] works showing evidence of stabilization of superconductivity in low dimensional systems due to the presence of tunnelling contacts with normal metallic leads, which suppress of fluctuations of the superconducting order-parameter.

In this article we study the effects of the screening of the Coulomb interaction in a quasi-1D superconductor by the presence of a metal nearby (cf. Fig. 1). To that end, we derive a phase-only action of the coupled system valid at low energies, which amounts to performing an RPA-approximation of the interacting problem. We specify two experimentally relevant geometries, namely: a) a 1D and b) a 2D electron gas (1DEG and 2DEG, respectively) in the diffusive limit. Our results show a rich behavior of the 1D plasma mode in the wire due to screening effects, and point towards the importance of the dimensionality of the screening metal. In particular in the case of screening provided by a 1DEG important frictional effects are observed in the superconductor due to the capacitive coupling, and in the limit $q \to 0$ and $T \to 0$ phase-coherence is destroyed and the wire shows finite resistivity. In contrast, for a wire screened by a 2DEG, friction and dissipation vanish in the limit $q \to 0$, and the wire is well described by the Luttinger liquid picture.

The article is divided as follows: in Sec. II we derive a general effective phase-only action for the complete superconductor-normal system, in Sec. III we present an analysis of the screening regimes at low energies for both the 1D and 2D geometries, Sec. IV is devoted to the study of the dissipative effects in the dynamical conductivity $\sigma(q,\omega)$ of the wire, and finally in Sec. V we summarize our findings and present a discussion. The details of the derivation of the low-energy effective action are given in the Appendices A and B.

II. MODEL

In this section we derive a general effective model which describes a clean superconductor capacitively coupled to a diffusive metal. We leave for Sec. III the specific analysis of the systems depicted in Fig. 1, representing a superconducting wire of length L and lateral dimensions $r_0 \ll \xi_0$ coupled to a diffusive metal placed at a distance d. The derivation of the model is standard[12,20,28,29] and here we only sketch the main steps. We refer the reader to the Appendix A and to the aforementioned references for details.

In the following we use the convention $\hbar = k_B = 1$. We begin our description with the microscopic action of the complete system

$$S = \int_0^\beta d\tau \sum_{a,\sigma} \int d\mathbf{r}\, \psi^*_{a,\sigma} \left(\partial_\tau - \mu\right) \psi_{a,\sigma} + \int_0^\beta d\tau\, H, \quad (1)$$

where $\beta = \frac{1}{T}$. The Grassmann field $\psi_{a,\sigma} \equiv \psi_{a,\sigma}(\mathbf{r},\tau)$ describes an electron in the superconductor for $a = s$ (normal metal for $a = n$) with spin projection σ at position $\mathbf{r} \equiv (x,y,z)$ and imaginary-time τ. The chemical potential $\mu = k_{\mathrm{F}}^2/2m$ is the Fermi energy in the normal state, with k_{F} the Fermi wavevector. The Hamiltonian H of the systems is

$$H = H^0_{\mathrm{s}} + H^0_{\mathrm{n}} + H_{\mathrm{int}}, \quad (2)$$

where

$$H^0_{\mathrm{s}} = \int d\mathbf{r} \sum_\sigma \frac{\left[\nabla\psi^\dagger_{s,\sigma}\right]\left[\nabla\psi_{s,\sigma}\right]}{2m} + U\ \bar{\psi}_{s\uparrow}\bar{\psi}_{s\downarrow}\psi_{s\downarrow}\psi_{s\uparrow}, \quad (3)$$

describes a translationally invariant, clean superconductor. Since we will not focus on the details of the pairing mechanism, here we assume a phenomenological local attractive interaction $U < 0$ which is responsible for (s-wave) pairing at $T < T_c$.

The normal metal is described by

$$H^0_{\mathrm{n}} = \int d\mathbf{r} \sum_\sigma \left\{ \frac{\left[\nabla\psi^\dagger_{n,\sigma}\right]\left[\nabla\psi_{n,\sigma}\right]}{2m} + \psi^\dagger_{n,\sigma} V_{\mathrm{i}} \psi_{n,\sigma} \right\}, \quad (4)$$

where $V_{\mathrm{i}} \equiv V_{\mathrm{i}}(\mathbf{r})$ represents the weak static impurity potential which provides a finite resitivity in the metal.

Finally, the interaction term of the whole system is given by

$$\begin{aligned} H_{\mathrm{int}} = &\frac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_2\ \hat{\rho}_s(\mathbf{r}_1)\, v(\mathbf{r}_1 - \mathbf{r}_2, 0)\, \hat{\rho}_s(\mathbf{r}_2) + \\ &+ \frac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_2\ \hat{\rho}_n(\mathbf{r}_1)\, v(\mathbf{r}_1 - \mathbf{r}_2, 0)\, \hat{\rho}_n(\mathbf{r}_2) + \\ &+ \int d\mathbf{r}_1 d\mathbf{r}_2\ \hat{\rho}_s(\mathbf{r}_1)\, v(\mathbf{r}_1 - \mathbf{r}_2, d)\, \hat{\rho}_n(\mathbf{r}_2), \end{aligned} \quad (5)$$

where we defined the electronic density operators $\hat{\rho}_a(\mathbf{r}) \equiv \sum_\sigma \psi^\dagger_{a,\sigma}(\mathbf{r})\,\psi_{a,\sigma}(\mathbf{r})$, and where the domain of integration of the variables $\mathbf{r}_1$ and $\mathbf{r}_2$ is constrained to the volume of the superconductor (if $a = s$) and the metal (if $a = n$). The interaction potential $v(\mathbf{r},z)$ is the microscopic long-range Coulomb interaction

$$v(\mathbf{r},z) = \frac{1}{\epsilon_{\mathrm{r}}} \frac{e^2}{\sqrt{r^2+z^2}},$$

where ϵ_{r} is the dielectric constant of the insulating medium between the metal and the superconductor.

The first step in the derivation of an effective low-energy model consists in decoupling the interaction terms

appearing in H_s^0 and $H_{\rm int}$ by the means of suitable Hubbard-Stratonovich transformations (HSTs). The repulsive Coulomb interaction $H_{\rm int}$ is more conveniently decoupled by expressing it in terms of the symmetric and antisymmetric density operators

$$\hat{\rho}_\pm(\mathbf{r}) \equiv \hat{\rho}_s(\mathbf{r}) \pm \hat{\rho}_n(\mathbf{r}). \quad (6)$$

With this definition, the interaction term [cf. Eq. (5)] compactly writes

$$H_{\rm int} = \frac{1}{2}\sum_{\nu=\pm}\int d\mathbf{r}_1 d\mathbf{r}_2\ \hat{\rho}_\nu(\mathbf{r}_1)\, v_\nu(\mathbf{r}_1-\mathbf{r}_2)\,\hat{\rho}_\nu(\mathbf{r}_2), \quad (7)$$

where we have defined

$$v_\nu(\mathbf{r}) \equiv \frac{v(\mathbf{r},0)+(\nu)\, v(\mathbf{r},d)}{2} \quad (\text{with } \nu=\pm). \quad (8)$$

The HSTs to decouple the long-range Coulomb and the Hubbard $U<0$ interactions are implemented by introducing the HS fields $\tilde{\rho}_\nu(\mathbf{r},\tau)$ in the particle-hole channel, and $\bar{\Delta}(\mathbf{r},\tau), \Delta(\mathbf{r},\tau)$ in the particle-particle channel, respectively (cf. Appendix A).

The next step in our derivation is to introduce an extra HS field $\rho_\nu(\mathbf{r},\tau)$ in order to decouple the quadratic term in $\tilde{\rho}_\nu(\mathbf{r},\tau)$, appearing in Eq. (A1). Then, it is easy to show that the field $\tilde{\rho}_\nu(\mathbf{r},\tau)$ can be formally integrated out, yielding a functional-delta function[30] $\delta\left[\hat{\rho}_\nu(\mathbf{r},\tau)-\rho_\nu(\mathbf{r},\tau)\right]$. As noted by De Palo *et al.*[20], this fact allows to interpret the new HS fields $\rho_\nu(\mathbf{r},\tau)$ as the *physical density* of the problem, expressed in our case in terms of the symmetric and antisymmetric collective modes.

At sufficiently low energies, amplitude fluctuations of the order parameter $\Delta(\mathbf{r},\tau)$ can be neglected, allowing to write $\Delta(\mathbf{r},\tau)=\Delta_0 e^{i\theta(\mathbf{r},\tau)}$, with a real constant Δ_0. The phase field $\theta(\mathbf{r},\tau)$ can be absorbed by a unitary transformation of the fermionic field

$$\psi_{s,\sigma}(\mathbf{r},\tau)\rightarrow\psi'_{s,\sigma}(\mathbf{r},\tau)=\psi_{s,\sigma}(\mathbf{r},\tau)\,e^{i\theta(\mathbf{r},\tau)/2}.$$

The derivation of the effective model proceeds with the integration of the fermionic fields $\psi_{a,\sigma}$, and by expanding the resulting bosonic action around the saddle-point in terms of the derivatives of $\theta(\mathbf{r},\tau)$ and the density fluctuations $\delta\tilde{\rho}_\nu(\mathbf{r},\tau), \delta\rho_\nu(\mathbf{r},\tau)$ [cf. Eqs. (A10) and (A11)]. This expansion amounts to performing an RPA-approximation of the interacting problem[20,31].

The last step is to integrate the auxiliary field $\delta\tilde{\rho}_\nu(\mathbf{r},\tau)$, which in the original representation of the density in terms $\delta\rho_s(\mathbf{r},\tau), \delta\rho_n(\mathbf{r},\tau)$ yields

$$\begin{aligned} S_{\rm eff} = &\int d\mathbf{r}d\tau\ \frac{i}{2}\partial_\tau\theta(\mathbf{r},\tau)\,\rho_s(\mathbf{r},\tau)+\\ &+\frac{1}{2}\int\prod_{i=1}^{2}d\mathbf{r}_i d\tau_i\ \left[\mathcal{D}(\mathbf{r}_1-\mathbf{r}_2,\tau_1-\tau_2)\times\right.\\ &\times\nabla\theta(\mathbf{r}_1,\tau_1)\,\nabla\theta(\mathbf{r}_2,\tau_2)+\\ &\left.+\boldsymbol{\delta\rho}^\dagger(\mathbf{r}_1,\tau_1)\,\boldsymbol{V}(\mathbf{r}_1-\mathbf{r}_2,\tau_1-\tau_2)\,\boldsymbol{\delta\rho}(\mathbf{r}_2,\tau_2)\right], \quad (9)\end{aligned}$$

where $\mathcal{D}(\mathbf{r},\tau)$ is the phase stiffness of the superconductor [cf. Eq. (A15)] and

$$\begin{aligned}\boldsymbol{\delta\rho}(\mathbf{r},\tau) &\equiv \begin{pmatrix}\delta\rho_s(\mathbf{r},\tau)\\ \delta\rho_n(\mathbf{r},\tau)\end{pmatrix},\\ \boldsymbol{V}(\mathbf{r},\tau) &\equiv \begin{pmatrix}\left[\chi_{0,s}(\mathbf{r},\tau)\right]^{-1} & 0\\ 0 & \left[\chi_{0,n}(\mathbf{r},\tau)\right]^{-1}\end{pmatrix}+\\ &+\delta(\tau)\begin{pmatrix} v(\mathbf{r},0) & v(\mathbf{r},d)\\ v(\mathbf{r},d) & v(\mathbf{r},0)\end{pmatrix},\end{aligned}$$

where we have used the notation $\left[\chi_{0,a}(\mathbf{r},\tau)\right]^{-1} \equiv \frac{1}{\beta V}\sum_{\mathbf{k},\omega_m} e^{i\mathbf{k}.\mathbf{r}-i\omega_m\tau}\chi_{0,a}^{-1}(\mathbf{k},\omega_m)$, where $\chi_{0,a}(\mathbf{k},\omega_m)$ is the bare density-density correlator (i.e., obtained with the Hamiltonian H_a^0), defined in Eqs. (A12) and (A13). Here we have used the notation in Fourier space $(\mathbf{k},\omega_m)$ with $\mathbf{k}$ the momentum and $\omega_m=\frac{2\pi m}{\beta}$ the bosonic Matsubara frequencies[31].

Note that at $T=0$ and in absence of quasiparticle excitations, the whole electronic density in the superconductor corresponds to the superfluid density. Consequently, the field $\delta\rho_s(\mathbf{r},\tau)$ physically represents the fluctuation of the Cooper-pair density at point $(\mathbf{r},\tau)$.

An interesting aspect of the effective action in Eq. (9) is that the first term (i.e., coupling between the total density of Cooper-pairs $\rho_s(\mathbf{r},\tau)$ and the phase-field $\theta(\mathbf{r},\tau)$) appears naturally as a consequence of the well-known number-phase commutation-relation $\left[\rho_s(\mathbf{r}),\theta(\mathbf{r}')\right]=i\delta(\mathbf{r}-\mathbf{r}')$ occurring in the superconducting groundstate[32].

Besides the contribution of soft modes, encoded in Eq. (9), in low-dimensional superconductors there are also stable topological excitations which contribute to the effective action. These are the well-known classical vortex (in 2D) and the phase slips (in 1D) excitations[32–34]. Focusing in the 1D case, a phase-slip is a region of size $\sim\xi_0$ where the order parameter temporarily vanishes, allowing the field $\theta(\mathbf{r},\tau)$ to perform a jump of $\pm2\pi n$ (with n integer) across it. For wires in the limit of very low superconducting stiffness, phase slips are an important source of momentum-unbinding, and a relevant contribution to the action in the RG-sense[11,12,35]. Indeed, it is believed that the eventual destruction of the superconducting state in isolated ultrathin wires occurs through the proliferation of quantum phase slips/anti phase slips pairs[12,13,16–18,35–37], in what constitutes the quantum analog in 1+1 dimensions to the classical Berezinskii-Kosterlitz-Thouless (BKT) transition in two space dimensions[38].

Note that our derivation does not account for the presence of phase-slips. Consequently, our results will only apply far from the BKT transition and far from the (non-superconducting) phase where the effect of phase slips dominates.

In the following we analyze the generic action of Eq.(9) for the different configurations of Fig. 1.

III. SCREENING REGIMES

A. Unscreened isolated wire

Let us first explore the instructive case of a superconducting wire ideally isolated from the environment. This situation corresponds to the normal metal placed infinitely far from the superconductor (i.e., $d \to \infty$), which results in the decoupling of their dynamics. For a very narrow superconducting wire with $r_0 \ll \xi_0$, the dependence of the fields $\theta(\mathbf{r},\tau), \delta\rho_s(\mathbf{r},\tau)$ on transverse dimensions can be neglected, reducing to $\{\theta(\mathbf{r},\tau), \delta\rho_s(\mathbf{r},\tau)\} \to \{\theta(\mathbf{x}), \delta\rho_s(\mathbf{x})\}$ where the compact notation $\mathbf{x} =\equiv (x,\tau)$ has been used. The effective action of the superconducting wire writes most conveniently in Fourier space

$$S_0^{\mathrm{w}} \simeq \frac{1}{2\beta L}\sum_{\mathbf{q}} \omega_m \theta(\mathbf{q})\rho_s^*(\mathbf{q}) + q^2\mathcal{D}(\mathbf{q})|\theta(\mathbf{q})|^2 +, + \frac{1+\chi_{0,s}(\mathbf{q})v(q,0)}{\chi_{0,s}(\mathbf{q})}|\rho_s(\mathbf{q})|^2, \quad (10)$$

where we have used the notation $\mathbf{q} \equiv (q, -\omega_m)$ (with q the momentum along the wire), and the property of real fields $\theta^*(\mathbf{q}) = \theta(-\mathbf{q})$, $\rho_s^*(\mathbf{q}) = \rho_s(-\mathbf{q})$. The Fourier transforms $\mathcal{D}(\mathbf{q})$ and $\chi_{0,s}(\mathbf{q})$ are defined as

$$\mathcal{D}(\mathbf{q}) = \int_0^\beta d\tau \int_0^L dx \int dydz\, e^{-i\mathbf{q}\mathbf{x}}\, \mathcal{D}(\mathbf{r},\tau), \quad (11)$$

$$\chi_{0,s}(\mathbf{q}) = \int_0^\beta d\tau \int_0^L dx \int dydz\, e^{-i\mathbf{q}\mathbf{x}}\, \chi_{0,s}(\mathbf{r},\tau). \quad (12)$$

At this point is relevant to calculate the Fourier transform of the Coulomb potential, which can be approximated as $v(\mathbf{r},z) \simeq \frac{e^2}{\epsilon_{\mathrm{r}}\sqrt{x^2+r_0^2+z^2}}$ (i.e., only dependent on the spatial coordinate x), and cut off at short distances by the radius r_0. Therefore we have

$$v(q,d) = \frac{2e^2}{\epsilon_{\mathrm{r}}} K_0\left(|q|\sqrt{r_0^2+d^2}\right), \quad (13)$$

where $K_0(\zeta)$ is the zeroth-order Bessel function, which verifies the limit $\lim_{\zeta\to 0} K_0(\zeta) \to -\ln\left(\frac{\zeta}{2}\right) - \gamma$, with γ the Euler gamma constant[39]. From the above Eq. (10) we obtain the phase-only action in the limit $\mathbf{q} \to 0$, by integration of the field $\rho_s(\mathbf{q})$

$$S_0^{\mathrm{w}} \simeq \frac{1}{2\beta L}\sum_{\mathbf{q}} \left[\frac{\omega_m^2}{\frac{8e^2}{\epsilon_{\mathrm{r}}}\ln\left(\frac{2}{|qr_0|}\right)} + q^2\mathcal{D}_0\right]|\theta(\mathbf{q})|^2, \quad (14)$$

where $\mathcal{D}_0 \equiv \lim_{\mathbf{q}\to 0} \mathcal{D}(\mathbf{q}) = \frac{\rho_s^{(0)}}{4m}$ (cf. Appendix B).

The minimization of the effective action Eq. (14) allows to obtain the equation of motion for the phase-field and to recover the dispersion-relation predicted for the 1D-plasma mode[10,11] upon analytical continuation to real frequencies $i\omega_m \to \omega + i0^+$

$$\omega^2(q) - \frac{8e^2}{\epsilon_{\mathrm{r}}}\mathcal{D}_0 q^2 \ln\left(\frac{2}{|qr_0|}\right) = 0. \quad (15)$$

Let us now concentrate on the superconducting properties of the wire. It is well-known that long-range order of the order parameter in 1D quantum systems is not possible, due to presence of strong quantum fluctuations and, strictly speaking, only quasi-long-range order, characterized by a slowly decreasing order-parameter correlation function

$$F(\mathbf{x}) \equiv \langle\Delta^*(\mathbf{x})\Delta(0)\rangle = \Delta_0^2 e^{-\frac{1}{2}\left\langle T_\tau[\theta(\mathbf{x})-\theta(0)]^2\right\rangle}, \quad (16)$$

can exist[11,40]. In the case of the isolated wire, the phase-correlation function calculated with the effective phase-only action Eq. (14) writes[11,41]

$$\left\langle T_\tau[\theta(\mathbf{x})-\theta(0)]^2\right\rangle = \frac{1}{\pi K}\left[\ln\left(\frac{\sqrt{x^2+u^2\tau^2\ln\tau}}{r_0}\right)\right]^{3/2}, \quad (17)$$

where $K \equiv \sqrt{\frac{\mathcal{D}_0\epsilon_{\mathrm{r}}}{8e^2}}$ and $u \equiv \sqrt{\frac{\mathcal{D}_0 8e^2}{\epsilon_{\mathrm{r}}}}$. As compared with the case of a 1D superconductor with short-range repulsive interactions[11], the phase correlator of Eq. (17) produces a relatively fast decrease of the order-parameter correlation function Eq. (16), as a consequence of the long-range Coulomb interaction, which is not completely screened in the 1D geometry. Consequently, density fluctuations are suppressed in the limit $\mathbf{q} \to 0$[41], and superconductivity, which benefits from fluctuations in the density, is suppressed.

A natural step to take in order to diminish the detrimental effects of the Coulomb interaction in the 1D geometry, is to screen it by the means of a metal placed nearby. This is the subject of the subsequent sections.

B. Screening by a diffusive metallic wire

We now concentrate on the system depicted in Fig. 1(a). For simplicity, we consider the case of two geometrically identical cylindrical wires. Extensions to other 1D geometries are straightforward. We assume that the normal metal is only one-dimensional with respect to density fluctuations $\rho_n(\mathbf{q})$ with spatial wavevector q satisfying the condition $qr_0 \ll 1$. Note that this condition does not necessarily imply that the normal wire is *electronically* 1D (i.e., it does not imply the existence of only one electronic conduction channel). Indeed, in what follows we assume a normal metal with a large number of channels $N_{\mathrm{ch}} \sim (k_{\mathrm{F}}r_0)^2 \gg 1$. This fact, together with the additional assumption of a very weak disorder potential, allows to neglect Anderson-localization effects (i.e.,

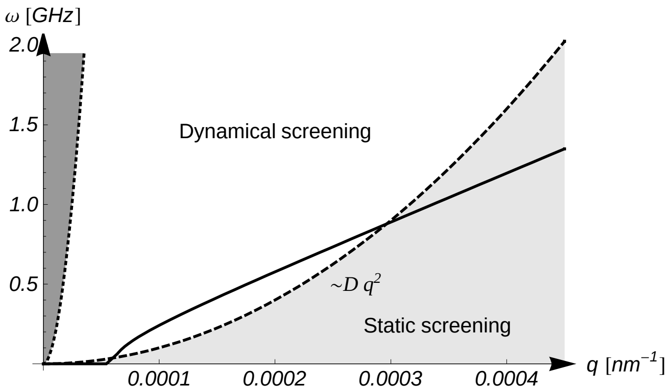

Figure 2: Screening regimes for a superconducting wire screened by a diffusive 1DEG. The curve $\omega = Dq^2$ separates the regime of static screening $\omega \ll Dq^2$ (light gray area) from that of dynamical screening $\omega \gg Dq^2$ (white area). The dispersion relation of the 1D plasma mode (thick solid line) is obtained from the solution of Eq. (20). For typical experimental values (cf. Table I), the dispersion relation crosses over from the static regime to the dynamical regime, and eventually the mode is completely damped. The unscreened regime of frequencies $\omega \gg Dq^2 \frac{2e^2}{\epsilon_r} \mathcal{N}^0_{n,1D} \ln \frac{2}{qr_0}$ corresponds to the dark gray area.

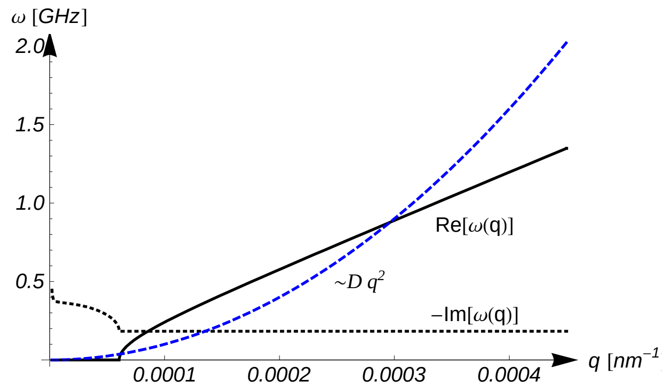

Figure 3: Real and imaginary components of the 1D plasma mode $\omega(q)$, obtained from the equation of motion of the action Eq. (20). The real part (solid line) gives the dispersion relation, while the imaginary part (dotted line) represents the damping of the mode. As in Fig. 2, the curves have been calculated for realistic experimental parameters (cf. Table I). The curve Dq^2 (blue dashed line) is shown as a reference.

$L \ll \xi_{\mathrm{wire}}$, where ξ_{wire} is the localization length in the diffusive normal wire).

In that case, Eq. (9) reduces to

$$S^{\mathrm{w}}_{(1)} \simeq \frac{1}{2\beta L} \sum_{\mathbf{q}} \Big[\omega_m \theta(\mathbf{q}) \rho^*_s(\mathbf{q}) + q^2 \mathcal{D}(\mathbf{q}) |\theta(\mathbf{q})|^2 + \\ + \boldsymbol{\rho}^\dagger(\mathbf{q}) \boldsymbol{V}(\mathbf{q}) \boldsymbol{\rho}(-\mathbf{q}) \Big], \tag{18}$$

where the subindex g in $S^{\mathrm{w}}_{(g)}$ indicates the effective dimensionality of the metal. The integration of the density modes $\rho_s(\mathbf{q})$ and $\rho_n(\mathbf{q})$ in the above expression allows to obtain the result

$$S^{\mathrm{w}}_{(1)} \simeq \frac{1}{2\beta L} \sum_{\mathbf{q}} \left\{ \frac{\omega^2_m}{4} \left[\frac{1 + \Big(\chi_{0,s}(\mathbf{q}) + \chi_{0,n}(\mathbf{q})\Big) v(q,0)}{\chi_{0,s}(\mathbf{q}) \left(1 + \chi_{0,n}(\mathbf{q}) v(q,0)\right)} + \right.\right. \\ \left. + \frac{\chi_{0,s}(\mathbf{q}) \chi_{0,n}(\mathbf{q}) \left(v(q,0)^2 - v(q,d)^2 \right)}{\chi_{0,s}(\mathbf{q}) \left(1 + \chi_{0,n}(\mathbf{q}) v(q,0)\right)} \right]^{-1} + \\ \left. + q^2 \mathcal{D}(\mathbf{q}) \right\} |\theta(\mathbf{q})|^2 . \tag{19}$$

In the following we focus on the experimentally relevant regime $d \approx r_0 \ll q^{-1}$. In that case the quantity $v(q,0)^2 - v(q,d)^2 \sim \left[\ln \frac{r_0}{d}\right]^2 \simeq 0$ drops from Eq. (19) and the expression simplifies to

$$S^{\mathrm{w}}_{(1)} \simeq \frac{1}{2\beta L} \sum_{\mathbf{q}} \left\{ \frac{\omega^2_m}{4} \frac{\chi_{0,s}(\mathbf{q}) \left[1 + \chi_{0,n}(\mathbf{q}) v(q,0)\right]}{1 + \left[\chi_{0,s}(\mathbf{q}) + \chi_{0,n}(\mathbf{q})\right] v(q,0)} + \\ + q^2 \mathcal{D}(\mathbf{q}) \right\} |\theta(\mathbf{q})|^2 . \tag{20}$$

For a weakly-disordered diffusive electron gas with elastic mean-free path l_e and scattering time $\tau_e = l_e / v_F$, where v_F is the Fermi velocity, the disorder-averaged density-density correlation function [cf. Eq. (A13)] at energies $|\omega_m| < \tau_e^{-1}$ and momentum $q < l_e^{-1}$ writes[42]

$$\chi_{0,n}(\mathbf{q}) \simeq 2\mathcal{N}^0_{n,1D} \frac{Dq^2}{Dq^2 + |\omega_m|}, \tag{21}$$

where $\mathcal{N}^0_{n,1D}$ is the 1D density of states at the Fermi energy in the normal metal, and $D = l_e^2/\tau_e$ is the diffusion constant in 1D. The factor 2 accounts for the spin degeneracy.

Note that the susceptibility $\chi_{0,n}(\mathbf{q})$ [cf. Eq. (21)] is non-analytical in the limit $\mathbf{q} \to 0$ for a normal diffusive metal. On the contrary, for the superconductor the presence of a gap in the excitation spectrum allows to obtain a well-defined limit $\lim_{\mathbf{q}\to 0} \chi_{0,s}(\mathbf{q}) \simeq \gamma \mathcal{N}^0_{s,1D}$, where $\mathcal{N}^0_{s,1D}$ is the linear density of states in the superconductor (in the normal state) at the Fermi level, and γ is a numerical coefficient of order 1 [cf. Eq. (B1)].

The plasma mode obtained from the equations of motion derived from Eq. (20) is plotted in Fig. 2 (thick solid line). Due to the complexity of the screening provided by the diffusive 1DEG, it is instructive to derive analytical expressions valid in the limiting cases of static (i.e., $|\omega_m| \ll Dq^2$) and dynamical (i.e., $Dq^2 \ll |\omega_m|$) screening.

1. Static screening limit $|\omega_m| \ll Dq^2$

This limit corresponds to the region $|\omega_m| \ll Dq^2$ (see light gray area in Fig. 2). In this case, the susceptibility in the normal metal can be approximated as $\chi_{0,n}(\mathbf{q}) \simeq 2\mathcal{N}^0_n \left(1 - \frac{|\omega_m|}{Dq^2}\right)$ [cf. Eq. (21)] . Then Eq. (20) can be

written as

$$S_{(1)}^{\mathrm{w}} \simeq \frac{1}{2\beta L}\sum_{\mathbf{q}}\left[\chi_s(0)\,\omega_m^2\left(1-\frac{\alpha|\omega_m|}{Dq^2}\right)+q^2\mathcal{D}_0\right]|\theta(\mathbf{q})|^2 \tag{22}$$

with $\chi_s(0) \equiv \frac{2\gamma\mathcal{N}_{n,1D}^0\mathcal{N}_{s,1D}^0}{\gamma\mathcal{N}_{s,1D}^0+2\mathcal{N}_{n,1D}^0}$ the effective static RPA-susceptibility of the wire and $\alpha \equiv \frac{\gamma\mathcal{N}_{s,1D}^0}{\gamma\mathcal{N}_{s,1D}^0+2\mathcal{N}_{n,1D}^0}$. Note that in the limit $\alpha \to 0$, the above action corresponds to a Luttinger liquid action with short-range interactions[11]. In the case of a 1D geometry of Fig. 1(a), the screening length is given by the distance d.

In the more general case of $\alpha > 0$, the term $\sim \frac{\alpha|\omega_m|}{Dq^2}$ introduces dissipation in the plasmon mode. From Eq. (22), the dispersion relation for the plasma-mode writes

$$-\omega^2(q)\left(1+i\frac{\alpha\omega(q)}{Dq^2}\right)-\frac{\mathcal{D}_0}{\chi_s(0)}q^2=0. \tag{23}$$

This equation holds provided the consistency condition $|\omega(q)| \ll Dq^2$ is verified (cf. solid line in Fig. 2). In Fig. 3 we show the solution of the above Eq. (23) as a function of q. Note that while $\mathrm{Re}[\omega(q)]$ follows an approximately linear dispersion relation, the imaginary part takes a constant value in the regime $|\omega(q)| \ll Dq^2$, meaning that the plasmon mode acquires a finite width, which in the perturbative limit $\alpha \to 0$ writes $\Gamma(q) \equiv -\mathrm{Im}[\omega(q)] \simeq \frac{\alpha\mathcal{D}_0}{2D\chi_s(0)}$ (cf. Fig. 3).

2. *Dynamic screening limit* $|\omega_m| \gg Dq^2$

For realisitic estimates of the experimental parameters (cf. Table I), our results indicate that the regime $|\omega_m| \gg Dq^2$ (white area in Fig. 2) is the most relevant for experimental studies on today's accesible wires[16–18]. Replacing Eq. (21) into Eq. (20) we note that if the condition

$$Dq^2 \ll |\omega_m| \ll \frac{2e^2}{\epsilon_{\mathrm{r}}}\mathcal{N}_{n,1D}^0 Dq^2\ln\frac{2}{qr_0}, \tag{24}$$

is fulfilled, the action in Eq. (20) can be approximated as

$$S_{(1)}^{\mathrm{w}} \simeq \frac{1}{2\beta L}\sum_{\mathbf{q}}\left[2\mathcal{N}_{n,1D}^0 Dq^2|\omega_m|+\mathcal{D}_0 q^2\right]|\theta(\mathbf{q})|^2. \tag{25}$$

The action Eq. (25) indicates that phase fluctuations show dissipative dynamics (encoded in the term $\sim q^2|\omega_m|$) as a consequence of the coupling to the dissipative processes in the 1DEG. In other words, the superconductor "inherits" the dissipation in the 1DEG through the Coulomb interaction.

Note that a term $\sim q^2|\omega_m|$ has been studied in the context of resistively shunted Josephson junctions arrays (RSJJAs)[14,43,44]. In that case, the term $\sim q^2|\omega_m|$ appears *in addition* to the dynamical term $\sim \omega_m^2$, which represents the effect of quantum fluctuations induced by the charging energy of the superconducting island[45]. As a result, dissipation turns out to be beneficial to superconductivity, through the quenching of phase fluctuations[43].

However, in our case, the form of the action in Eq. (25) is qualitatively different, since the term $\sim \omega_m^2$ is *absent* from the action (actually, it is the dynamical term *itself* which becomes a contribution $\sim q^2|\omega_m|$). This has *detrimental* consequences for the superconductivity in the wire, as can be seen directly from the equation of motion for the field θ, which gives $\omega(q) \simeq -i\mathcal{D}_0/\left(D\mathcal{N}_{n,1D}^0\right)$, indicating that the original plasma mode is completely damped and vanishes in the limit $q \to 0$ (see Fig. 2). Indeed, expressing the action Eq. (18) in terms of the dual field[11] $\phi(\mathbf{x})$, defined as

$$\delta\rho_s(\mathbf{x}) \equiv -\frac{1}{\pi}\nabla\phi(\mathbf{x}), \tag{26}$$

we obtain the equivalent description

$$S_{(1)}^{\mathrm{w}} = \frac{1}{2\pi^2}\frac{1}{\beta L}\sum_{\mathbf{q}}|\phi(\mathbf{q})|^2\left\{\frac{\omega_m^2}{4}\frac{1}{\mathcal{D}_0}+ +q^2\frac{1+[\chi_{0,s}(\mathbf{q})+\chi_{0,n}(\mathbf{q})]\,v(q,0)}{\chi_{0,s}(\mathbf{q})\,[1+\chi_{0,n}(\mathbf{q})\,v(q,0)]}\right\}.$$

In the regime of Eq. (24), we can approximate the action by

$$\begin{aligned}S_{(1)}^{\mathrm{w}} &\simeq \frac{1}{2\pi^2}\frac{1}{\beta L}\sum_{\mathbf{q}}\left\{\frac{\omega_m^2}{4}\frac{1}{\mathcal{D}_0}+ +q^2\frac{\left[\gamma\mathcal{N}_{s,1D}^0+2\mathcal{N}_{n,1D}^0\frac{Dq^2}{|\omega_m|}\right]v(q,0)}{2\gamma\mathcal{N}_{n,1D}^0\mathcal{N}_{s,1D}^0\frac{Dq^2}{|\omega_m|}v(q,0)}\right\}|\phi(\mathbf{q})|^2,\\ &= \frac{1}{2\pi^2}\frac{1}{\beta L}\sum_{\mathbf{q}}\left\{\frac{|\omega_m|}{2\mathcal{N}_{n,1D}^0 D}+\frac{\omega_m^2}{4\mathcal{D}_0}+\frac{q^2}{\gamma\mathcal{N}_{s,1D}^0}\right\}|\phi(\mathbf{q})|^2,\end{aligned}$$

which shows that the term $\sim q^2|\omega_m|$ in Eq. (25) translates into a relevant term $\sim |\omega_m|$ (in the RG-sense) when expressed in terms of $\phi(\mathbf{q})$. Another way to see this detrimental effect is through the order-parameter correlation function $F(\mathbf{x}) = \Delta_0^2 e^{-\frac{1}{2}\left\langle T_\tau[\theta(\mathbf{x})-\theta(0)]^2\right\rangle}$ [cf. Eq. (16)], which vanishes due to the infrared divergence of the phase-correlator $\left\langle T_\tau[\theta(\mathbf{x})-\theta(0)]^2\right\rangle \equiv \frac{2}{\beta L}\sum_{\mathbf{q}}\frac{1-\cos\mathbf{q.x}}{2\mathcal{N}_n^0 Dq^2|\omega_m|+\mathcal{D}_0 q^2} \to \infty$.

Only at high-frequencies $\frac{2e^2}{\epsilon_{\mathrm{r}}}\mathcal{N}_n^0 Dq^2\ln\frac{2}{qr_0} \ll |\omega_m|$ (cf. dark gray area in Fig. 2), and provided Eq. (21) is still valid, or in the limit of very low electronic density of states in the 1DEG, we recover the action of Eq. (14) describing again unscreened plasma modes. Physically, at such high frequencies the response $\chi_{0,n}(\mathbf{q})$ of the 1DEG vanishes and the superconducting wire is effectively unscreened.

$r_0 \simeq d$	L	$\mathcal{D}_0$	$\mathcal{N}^0_{s,1D} \simeq \mathcal{N}^0_{n,1D}$	Δ_0	D	$\epsilon_{\rm r}$	$w_{\rm film}$	$k^{2D}_{\rm TF}$	$\mathcal{N}^0_{n,2D}$
10 nm	100 μm	8.6 $10^{35}\,\frac{1}{\rm kg.m}$	$10^{29}\,\frac{1}{\rm m.J}$	1 K	0.01 $\frac{\rm m^2}{\rm s}$	1	100 nm	1 nm^{-1}	$10^{38}\,\frac{1}{\rm m^2.J}$

Table I: Parameters used in the calculations. Order-of-magnitude estimations of r_0 and L have been extracted from experiments on superconducting aluminium wires with coherence length estimated as $\xi_0 \sim 100$ nm [cf. Ref. 18].

C. Screening by a diffusive metallic film

Now we focus our attention on the system of Fig. 1(b), which represents a superconducting wire coupled to a normal diffusive film of width $w_{\rm film}$. In this case, the effective action Eq. (9) writes in Fourier space

$$S^{\rm w}_{(2)} \simeq \frac{1}{2\beta L}\sum_{\mathbf{q}}\Big\{\omega_m\theta^*(\mathbf{q})\,\rho_s(\mathbf{q}) + q^2\mathcal{D}(\mathbf{q})\,|\theta(\mathbf{q})|^2 + \left[\chi^{-1}_{0,s}(\mathbf{q}) + v(q,0)\right]|\rho_s(\mathbf{q})|^2\Big\} + \frac{1}{2\beta L L_\perp}\sum_{\mathbf{q},k}\Big\{\left[\chi^{-1}_{0,n}(\mathbf{q},k) + v(q,k,0)\right]|\rho_n(\mathbf{q},k)|^2 + v(q,k,d)\left[\rho^*_s(\mathbf{q})\,\rho_n(\mathbf{q},k) + \rho_s(\mathbf{q})\,\rho^*_n(\mathbf{q},k)\right]\Big\}, \tag{27}$$

where the Coulomb interaction is [compare to Eq. (13)]

$$v(q,k,d) = \frac{2\pi e^2}{\epsilon_{\rm r}}\frac{e^{-\sqrt{q^2+k^2}\,d}}{\sqrt{q^2+k^2}}.$$

In this case, the presence of the superconducting wire breaks the translational symmetry of the system in the direction perpendicular to the wire. Consequently, the perpendicular momentum k in the plane is not conserved and the Coulomb interaction couples the density modes in the wire with momentum q to all the modes in the plane with momentum k. As before, in order to obtain an effective model for the phase field, we must integrate over the density fields $\rho_n(\mathbf{q},k)$ and $\rho_s(\mathbf{q})$, which yields

$$S^{\rm w}_{(2)} \simeq \frac{1}{2}\frac{1}{\beta L}\sum_{\mathbf{q}}\Big\{\frac{\omega_m^2}{4}\frac{\chi_{0,s}(\mathbf{q})}{1+\chi_{0,s}(\mathbf{q})\left[v(q,0)-v_{\rm eff}(\mathbf{q})\right]} + q^2\mathcal{D}(\mathbf{q})\Big\}|\theta(\mathbf{q})|^2, \tag{28}$$

where $v_{\rm eff}(\mathbf{q})$ is an effective 1D-potential encoding all the screening properties of the diffusive film

$$v_{\rm eff}(\mathbf{q}) \equiv \frac{1}{L_\perp}\sum_k \frac{v(q,k,d)^2\,\chi_{0,n}(\mathbf{q},k)}{1+v(q,k,0)\,\chi_{0,n}(\mathbf{q},k)}. \tag{29}$$

In this case, the susceptibility at low energies writes[42]

$$\chi_{0,n}(\mathbf{q},k) \simeq 2\mathcal{N}^0_{n,2D}\frac{D\left(q^2+k^2\right)}{D\left(q^2+k^2\right)+|\omega_m|}, \tag{30}$$

where $\mathcal{N}^0_{n,2D}$ is the 2D density of states at the Fermi energy in the normal metal. Here again, we neglect Anderson-localization effects in the metal by assuming that the length of the wire is $L \ll \xi_{\rm film}$, where $\xi_{\rm film}$ is the localization length in the film.

In what follows, we derive analytical expressions for the model in the limiting cases of static and dynamic regimes.

1. Static screening limit $|\omega_m| \ll Dq^2$

This region corresponds to the ligh gray area in Fig. (4). A relevant length scale which naturally appears is the 2D Thomas-Fermi screening length, $\lambda^{\rm 2D}_{\rm TF} = \frac{\epsilon_{\rm r}}{2e^2\mathcal{N}^0_{n,2D}}$, beyond which the long-range Coulomb potential is completely screened. This quantity defines the Thomas-Fermi wavevector $k_{\rm TF} = \frac{4\pi e^2\mathcal{N}^0_{n,2D}}{\epsilon_{\rm r}}$. In the experimentally relevant limit $k_{\rm TF}d \gg 1$, the effective potential $v_{\rm eff}(\mathbf{q})$ reduces to

$$v_{\rm eff}(\mathbf{q}) \simeq \frac{2e^2}{\epsilon_{\rm r}}\left[K_0(2qd) - \frac{\pi}{2}\frac{|\omega_m|}{Dqk_{\rm TF}}\right]. \tag{31}$$

The first and second term in the above expression are consistent with the static and dissipative contributions, respectively, to the effective screened interaction obtained for a Tomonaga-Luttinger liquid electrostatically coupled to a diffusive 2DEG (cf. Ref. 23). The static screening provided by the 2DEG [first term in Eq. (31)] cuts the logarithmic divergence of the bare intrawire Coulomb interaction $v(q,0)$. The relationship between the second term in Eq. (31) and the dissipative contribution $\sim q\,|\omega_m|$ in Ref. 23 can be made explicit with the introduction of the field $\phi(\mathbf{x})$, defined in Eq. (26).

In the limit $qd \to 0$ (with $|\omega_m| \ll Dq^2$), the effective potential Eq. (31) can be further simplified to $v_{\rm eff}(\mathbf{q}) \simeq \frac{2e^2}{\epsilon_{\rm r}}\left[\ln\left(\frac{1}{qd}\right) - \frac{\pi}{2}\frac{|\omega_m|}{Dqk_{\rm TF}}\right]$, and when replaced in Eq. (28) yields

$$S^{\rm w}_{(2)} \simeq \frac{1}{2\beta L}\sum_{\mathbf{q}}\left[\frac{\omega_m^2}{4}\chi_s(0) - \frac{\pi e^2\chi_s^2(0)\,|\omega_m|^3}{Dk_{\rm TF}q} + q^2\mathcal{D}_0\right]|\theta(\mathbf{q})|^2,$$

where $\chi_s(0) \simeq \chi_{0,s}(0)\left[1+\frac{2e^2}{\epsilon_{\rm r}}\chi_{0,s}(0)\ln(2d/r_0)\right]^{-1}$ is the RPA susceptibility of the wire.

Note that in the limit $D \to \infty$ (no dissipation in the normal metallic film), we recover again the action of a Tomonaga-Luttinger liquid with short-range interactions, with plasma modes obeying a linear dipersion relation[11]. In the more general case of a finite D, we have the relation dispersion

$$\omega^2(q)\,\chi_s(0) + i\frac{e^2}{\epsilon_\mathrm{r}}\frac{\pi\omega^3(q)\,\chi_s^2(0)}{Dqk_\mathrm{TF}} + q^2\mathcal{D}_0 = 0, \tag{32}$$

which describes plasma modes with approximately linear dispersion relation, and with a width $\Gamma(q) = -\mathrm{Im}\left[\omega(q)\right] \sim \frac{\pi e^2\mathcal{D}_0}{2\epsilon_\mathrm{r}Dk_\mathrm{TF}\chi_{s,0}(0)}q$. Note that this result only applies in the limit $|\omega(q)| \ll \left\{Dq^2, \frac{Dqk_\mathrm{TF}\epsilon_\mathrm{r}}{\pi e^2\chi_s(0)}\right\}$, and eventually breaks down in the limit $q \to 0$, meaning that this is not the relevant regime at low energies.

2. *Dynamical screening limit $|\omega_m| \gg Dq^2$*

With realistic estimates for the experimental parameters (cf. Table I), the regime $|\omega_m| \gg Dq^2$ is the most relevant in practical realizations. In this regime (white area in Fig. 4), the effective potential $v_\mathrm{eff}(\mathbf{q})$ in Eq. (29) can be approximated as

$$v_\mathrm{eff}(\mathbf{q}) \approx \frac{2e^2}{\epsilon_\mathrm{r}}\left[f(2k_\mathrm{TF}d) - f\left(\frac{2|\omega_m|d}{Dk_\mathrm{TF}}\right)\right]$$

where we have defined $f(z) \equiv -e^z\mathrm{Ei}(-z)$, with $\mathrm{Ei}(x)$ the exponential integral function[39]. If the additional condition $|\omega_m| \ll \frac{Dk_\mathrm{TF}}{d}$ holds, the effective potential can be further simplified to

$$v_\mathrm{eff}(\mathbf{q}) \simeq \frac{2e^2}{\epsilon_\mathrm{r}}\ln\left(\frac{2|\omega_m|}{Dk_\mathrm{TF}}d\right).$$

Using this expression and Eq. (13), the phase-only action of Eq. (28) writes

$$S_{(2)}^\mathrm{w} \simeq \frac{1}{2\beta L}\sum_\mathbf{q}\left\{\frac{\omega_m^2}{4}\frac{\chi_{0,s}(0)}{1+\frac{2e^2}{\epsilon_\mathrm{r}}\chi_{0,s}(0)\ln\left(\frac{Dk_\mathrm{TF}}{dr_0 q|\omega_m|}\right)} + q^2\mathcal{D}_0\right\}|\theta(\mathbf{q})|^2,$$

resulting in the equation of motion (in the limit $\mathbf{q} \to 0$)

$$-\frac{\omega^2}{4} + q^2\frac{2e^2}{\epsilon_\mathrm{r}}\mathcal{D}_0\left(\ln\left|\frac{Dk_\mathrm{TF}}{dr_0 q\omega}\right| + i\frac{\pi}{2}\right) = 0. \tag{33}$$

In Fig. 4 we show (solid line) the dispersion relation obtained from Eq. (33) (i.e., real component of $\omega(q)$). Contrarily to the case studied in Sec. III B, the resulting plasma mode is not damped in the limit $q \to 0$, i.e., a dispersive real component survives. In order to investigate the dynamics of the phase-field at low-energies, we study

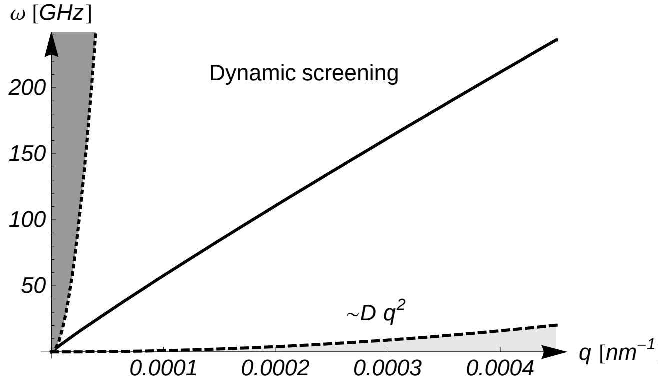

Figure 4: Screening regimes for a superconducting wire capacitively coupled to a diffusive 2DEG. The dispersion relation (solid line) results from Eq. (33), valid in the regime $Dq^2 \ll |\omega_m| \ll \frac{Dk_\mathrm{TF}}{d}$. As in Fig. 2, the dashed line $\omega = Dq^2$ separates the regime of static (light gray area) from that of dynamic (white area) screening. For the same parameters as in Fig. 2, the plasma mode crosses over between the two regimes at higher frequencies. The unscreened region corresponds to the dark gray area.

the ratio $\mathrm{Im}\left[\omega(q)\right]/\mathrm{Re}\left[\omega(q)\right]$ vs. q (cf. Fig. 5). From Eq. (33) it is possible to show that in the limit $q \to 0$

$$\frac{\mathrm{Im}\left[\omega(q)\right]}{\mathrm{Re}\left[\omega(q)\right]} \simeq \frac{\pi}{4}\left[\ln\left(\frac{Dk_\mathrm{TF}}{q^2 dr_0\sqrt{\frac{8e^2}{\epsilon_\mathrm{r}}\mathcal{D}_0}}\right)\right]^{-1}, \tag{34}$$

meaning that the width of the plasma mode decreases at low energies, resulting in a well-defined excitation. Note the difference with respect to the screening provided by a 1DEG, where the damping of the plasmon was complete in the limit $q \to 0$. The origin of this difference lies in the additional degree of freedom k (momentum perpendicular to the wire), which smears (upon integration) the dependence on the damping factor $|\omega_m|$ in the susceptibility [cf. Eq. (30)] .

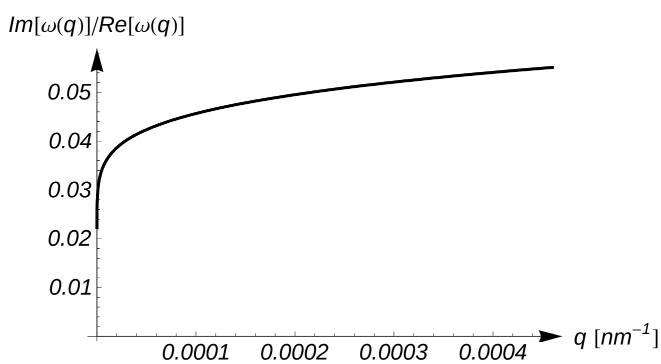

Figure 5: Ratio $\frac{\mathrm{Im}[\omega(q)]}{\mathrm{Re}[\omega(q)]}$ vs q for typical experimental parameters (cf. Table I). The plasma mode becomes better defined as $q \to 0$.

In the regime of frequencies $\frac{Dk_\mathrm{TF}}{d} \ll |\omega_m| \ll Dk_\mathrm{TF}^2$,

the effective potential $v_{\mathrm{eff}}(\mathbf{q})$ writes

$$v_{\mathrm{eff}}(\mathbf{q}) \approx \frac{2e^2}{\epsilon_{\mathrm{r}}}\frac{1}{2k_{\mathrm{TF}}d}\left[1-\frac{Dk_{\mathrm{TF}}^2}{|\omega_m|}\right].$$

and the phase only action is

$$S_{(2)}^{\mathrm{w}} \simeq \frac{1}{2}\frac{1}{\beta L}\sum_{\mathbf{q}}\left\{\frac{\omega_m^2}{4}\frac{\chi_{0,s}(0)}{1+\frac{2e^2}{\epsilon_{\mathrm{r}}}\chi_{0,s}(0)\left[\ln\left(\frac{2}{qr_0}\right)+\frac{Dk_{\mathrm{TF}}}{2d|\omega_m|}\right]}+ + q^2\mathcal{D}_0\right\}|\theta(\mathbf{q})|^2.$$

In this limit, the equation of motion of the field $\theta(\mathbf{q})$ is

$$-\omega^2(q)+q^2\mathcal{D}_0\frac{8e^2}{\epsilon_{\mathrm{r}}}\left[\ln\left(\frac{2}{qr_0}\right)+\frac{iDk_{\mathrm{TF}}}{2d\omega(q)}\right]=0.$$

Note that in this regime, the dissipative effects are even weaker and the dispersion relation ressembles that of the (unscreened) Mooij-Schön mode. Eventually in the limit $|\omega_m| \gg Dk_{\mathrm{TF}}^2$, the response $\chi_{0,n}(\mathbf{q})$ of the 2DEG vanishes and the wire is effectively in the unscreened regime where the Mooij-Schön plasma mode of Eq. (15) is fully recovered.

IV. DISSIPATIVE EFFECTS IN THE DYNAMIC CONDUCTIVITY

In this section we study the consequences of thedissipative effects on the dynamic conductivity of the wire $\sigma(q,\omega)$, i.e., the ratio between the current density and the local electric field $j(q,\omega)=\sigma(q,\omega)E(q,\omega)$. This quantity is of interest because its real part $\mathrm{Re}[\sigma(q,\omega)]$ provides information on the dissipation and absorption properties, which result in our case from the friction mediated by the Coulomb interaction[46].

The response of the system to an external electromagnetic field can be obtained by the means of the minimal coupling $-i\nabla \to -i\nabla - \frac{e}{c}A$ in the microscopic Hamiltonian Eq. (3). For a superconducting wire at $T=0$ and in absence of quasiparticle excitations, the total current density is given by

$$\begin{aligned} j(\mathbf{x}) &= j_p(\mathbf{x})+j_d(\mathbf{x}),\\ j_p(\mathbf{x}) &= \frac{2e}{c}\mathcal{D}_0\nabla\theta(\mathbf{x}),\\ j_d(\mathbf{x}) &= -\left(\frac{2e}{c}\right)^2\mathcal{D}_0A(\mathbf{x}),\end{aligned}$$

where j_p and j_d are, respectively, the paramagnetic and diamagnetic contributions to the current density. The linear response to an applied electromagnetic field is given by the current-current susceptibility of the wire

$$\chi_{jj}(\mathbf{q}) = \left.\frac{\delta\ln Z}{\delta A_{\mathbf{q}}\delta A_{-\mathbf{q}}}\right|_{A=0} = \langle j_p(\mathbf{q})\,j_p(-\mathbf{q})\rangle - \mathcal{D}_0\left(\frac{2e}{c}\right)^2.$$

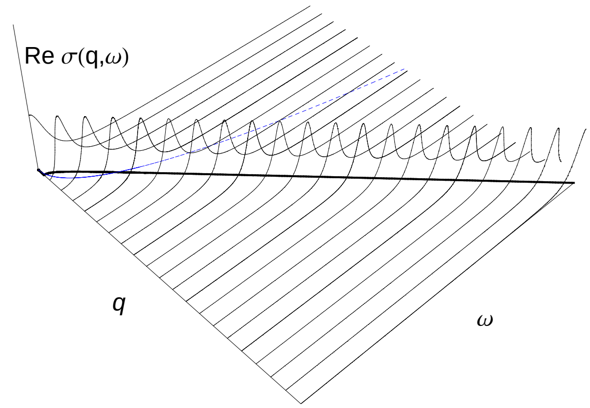

Figure 6: Dynamic conductivity $\mathrm{Re}\,\sigma(q,\omega)$ of a superconducting wire dynamically screened by a diffusive 1DEG. The plasma mode, which is better defined at high energy and momentum becomes completely damped in the limit $\{\omega,q\}\to 0$ by the effects of the dissipative environment.

The conductivity is in turn related to the current-current susceptibility by the relation[11,31] $\sigma(\mathbf{q}) = -\frac{\chi_{jj}(\mathbf{q})}{\omega_m}$, upon analytical continuation to real frequencies $\sigma(\mathbf{q})|_{i\omega_m\to\omega+i\delta}=\sigma(q,\omega)$. In terms of the phase field $\theta(\mathbf{q})$, the conductivity reads

$$\sigma(\mathbf{q}) \equiv -\left(\frac{2e}{c}\right)^2\left[-\frac{\mathcal{D}_0}{\omega_m}+\frac{\mathcal{D}_0^2q^2}{\omega_m}\langle\theta(\mathbf{q})\,\theta(-\mathbf{q})\rangle\right]. \quad (35)$$

Let us first study the response of an ideally isolated wire (cf. Sec. III A) to the electromagnetic field. At $T=0$ we obtain

$$\mathrm{Re}[\sigma(q,\omega)] = \frac{\pi}{2}\mathcal{D}_0\left(\frac{2e}{c}\right)^2\delta(\omega-\omega(q)), \quad (36)$$

where $\omega(q)=\sqrt{\frac{8e^2}{\epsilon_{\mathrm{r}}}\ln\left(\frac{2}{|qr_0|}\right)q^2\mathcal{D}_0}$ is the energy of the Mooij-Schön plasmon [cf. Eq. (15)]. The real part of the conductivity tells us that the system absorbs energy at the frequency $\omega=\omega(q)$, which in this case are well-defined excitations (i.e., delta-functions). Note that in the limit $q\to 0$, Eq. (36) allows to recover the Drude peak at $\omega=0$, which is expected for a superconductor[11,31].

Let us now consider the case of a wire in the proximity to an electron gas. We first study the case of screening by a diffusive 1DEG (cf. Sec. III B 2), where the effects of dissipation are at their strongest. Using the action of Eq. (20) to evaluate the formula of the conductivity [cf.

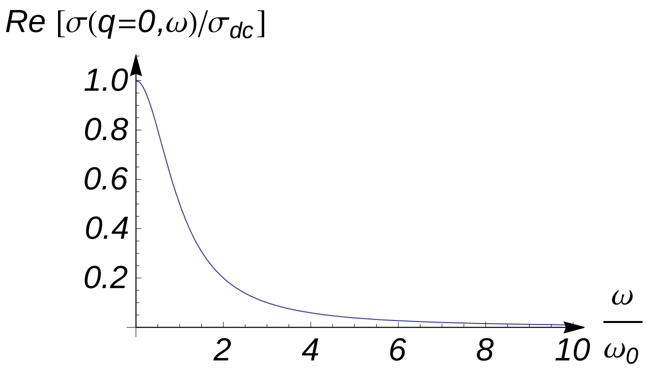

Figure 7: dc-conductivity $\sigma(\omega) = \mathrm{Re}[\sigma(q=0,\omega)]$ of a superconducting wire screened by a diffusive 1DEG. The values on the axis are normlized to $\sigma_{\rm dc} = \sigma(\omega=0) = \left(\frac{2e}{c}\right)^2 2D\mathcal{N}^0_{n,1D}$ and $\omega_0 \equiv \frac{\mathcal{D}_0}{2\mathcal{N}^0_{n,1D}D}$. The plasma mode at finite q, is completely damped in the limit $q=0$ (cf. Fig. 6).

Eq. (35)] we obtain the expression (valid at $T=0$)

$$\mathrm{Re}\left[\sigma(q,\omega)\right] = \mathcal{D}_0^2\left(\frac{2e}{c}\right)^2\frac{q^2}{\omega}\mathrm{Im}\left[q^2\mathcal{D}_0 - \frac{(\omega+i0^+)^2}{4}\times \times\frac{\chi_{0,s}(0)\left[1+\chi^{\rm ret}_{0,n}(q,\omega)\,v(q,0)\right]}{1+\left[\chi_{0,s}(0)+\chi^{\rm ret}_{0,n}(q,\omega)\right]v(q,0)}\right]^{-1}, \tag{37}$$

where $\chi^{\rm ret}_{0,n}(q,\omega) \equiv \lim_{\delta\to0^+}\left[\chi_{0,n}(\mathbf{q})\right]_{i\omega_m\to\omega+i\delta}$ is the (disorder-averaged) retarded density-density correlation function in the 1DEG. In Fig. 6 we show the result for $\mathrm{Re}[\sigma(q,\omega)]$ of Eq. (37) in the plane $q-\omega$. The dispersion relation $\mathrm{Re}[\omega(q)]$ vs. q (thick solid line in the bottom plane) was calculated numerically from Eq. (20) and corresponds to the same plot of Fig. 2. As mentioned before, the absorption peaks of $\mathrm{Re}[\sigma(q,\omega)]$ are centered at the frequency $\mathrm{Re}[\omega(q)]$ of the plasma mode. The curve Dq^2 (blue dotted line) is plotted in the bottom $q-\omega$ plane to visualize the different screening regimes. Note that the dissipative effects in the normal wire (encoded in a finite value of the diffussion constant D) are manifested in this figure through the finite width $\Gamma(q) \simeq -\mathrm{Im}[\omega(q)]$ of the plasmon peaks. Note in addition that the constant width $\Gamma(q)$ in the regime $|\omega(q)| \gg Dq^2$ is consistent with the result for $\mathrm{Im}[\omega(q)]$ of Fig. 3.

As $q\to0$, the plasmon peak merges smoothly into the dc-conductivity value $\sigma_{\rm dc} = \left(\frac{2e}{c}\right)^2 2D\mathcal{N}^0_{n,\rm 1D}$, which exactly corresponds to the dc-conductivity of the 1DEG (cf. Fig. 7). This expression is obtained by replacing the expression of the action of Eq. (25) in the general expression of the conductivity Eq. (35). Physically, this means that the Coulomb interaction produces friction in the superconductor through the dissipation existing the 1DEG. It also indicates that the original plasma mode is no longer a well-defined excitation of the system, and that the electromagnetic environment have profound consequences in the excitation spectrum of the 1D superconductor.

As we mentioned before, far from the BKT quantum critical point, phase slips are an irrelevant perturbation (in the RG-sense). In the case of Luttinger liquids with short-range interactions, the perturbative effect of phase-slips generates a power-law resistivity $\varrho \sim T^\nu$, with ν a positive exponent[35]. Although we have neglected the perturbative effect of topological excitations in our formalism, the fact that a finite resistivity at $T=0$ appears in the superconducting wire indicates that their effect in the conductivity might be negligible as compared to those induced by dissipation in the electron gas.

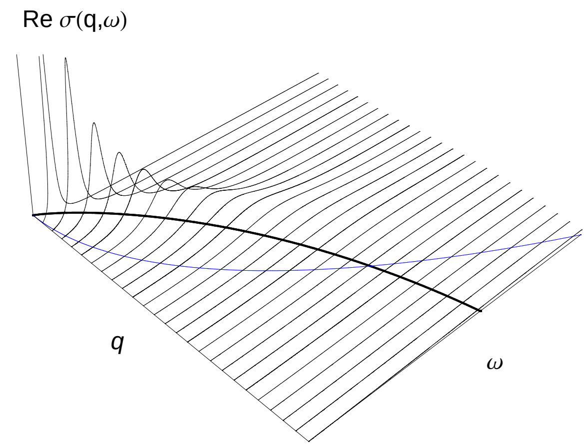

Figure 8: Dynamic conductivity $\mathrm{Re}[\sigma(q,\omega)]$ of a superconducting wire dynamically screened by a diffusive 2DEG. The plasma mode is completely damped at high energy and momentum, but in the limit $\{\omega, q\}\to0$ the effects of dissipation vanish.

In the case of screening by a diffusive 2DEG, the expression of the conductivity is given by the expression (valid at $T=0$)

$$\mathrm{Re}\left[\sigma(q,\omega)\right] = \mathcal{D}_0^2\left(\frac{2e}{c}\right)^2\frac{q^2}{\omega}\mathrm{Im}\left[q^2\mathcal{D}_0 - \frac{(\omega+i0^+)^2}{4}\times \times\frac{\chi_{0,s}(0)}{1+\chi_{0,s}(0)\left[v(q,0)-v^{\rm ret}_{\rm eff}(q,\omega)\right]}\right]^{-1},$$

where $v^{\rm ret}_{\rm eff}(q,\omega) \equiv \lim_{\delta\to0^+}\left[v_{\rm eff}(\mathbf{q})\right]_{i\omega_m\to\omega+i\delta}$. Our main results in this case are presented in Fig. 8. Contrarily to the case of Fig. 6, the plasmon peaks are worse defined at high energies, while at low energies the width of the peak centered at $\mathrm{Re}[\omega(q)]$ decreases and eventually vanishes in the limit $q\to0$, in agreement with Eq. (34) and Fig. (5). Eventually, the plasmon peak merges into the superconducting Drude peak at $\omega=0$.

The presence of an additional degree of freedom (i.e., momentum in the plane perpendicular to the wire) is of central importance to understand the vanishing of dissipation. Indeed, even in the dynamical screening regime

$Dq^2 \ll |\omega_m|$ for which one would naively think that dissipation effects are dominant, the presence of perpendicular wavevectors k satisfying the condition $|\omega_m| \ll Dk^2$ make the dissipative processes less important. Note in addition that this condition is more easily satisfied in the limit $|\omega_m| \to 0$. These qualitative phase-space considerations allow to understand the behavior of the effective 1D potential $v_{\text{eff}}(\mathbf{q})$ of Eq. (29), which produces a weaker (i.e., logarithmic) dependence on the term $|\omega_m|$ encoding the dissipation. The net result is that the 1D plasma modes are better defined in the limit $q \to 0$ and the frictional effects vanishes strictly in the thermodynamical limit $L \to \infty$.

V. DISCUSSION AND SUMMARY

In this article we have studied the effects of the local electromagnetic environment, provided by the presence of a non-interacting electron gas, on the low-energy physics of a superconducting wire. In particular, we have focused on the derivation of an effective phase-only action, starting from the microscopic Hamiltonian of the system. We make extensive use of the path-integral formalism, which enables to decouple the superconducting and long-range Coulomb interactions by the means of Hubbard-Stratonovich fields, and to expand the resulting action in terms of quadratic deviations of these fields around the saddle-point (i.e., Gaussian fluctuations). This treatment is equivalent to performing the so-called RPA-approximation of the interacting problem[31]. We have studied two cases in particular, namely, the screening provided by a diffusive 1DEG, and a diffusive 2DEG placed at a distance $d \simeq r_0$ from the wire, with r_0 its radius. This would be the relevant situation in practical realizations in, e.g., superconductor/normal heterostructures made by the means of the ferroelectric field-effect in Nb-doped $SrTiO_3$ layers[47], or in electrically controlled $LaAlO_3/SrTiO_3$ interfaces[48,49].

It is of interest to put our results in the context of other works dealing with electrostatically coupled 1D systems. Among these, the Coulomb drag effect[50], where a finite current I_1 is driven in one (the "active") system, and a finite voltage V_2 is induced in the other ("passive" system), has received a great deal of attention both theoretically[51–55] and experimentally[56–58]. Although closely related, the focus of our work is on the equilibrium properties of the wire.

From the theoretical point of view, our work differs from the usual Tomonaga-Luttinger liquid description of a purely 1D (i.e., one electronic conduction channel) conductor, where the main mechanism of momentum transfer is backscattering[23,53–55]. Indeed, it is worth to note that intra- and/or interwire backscattering effects are absent in clean wires with a large number of electronic channels, and this fact is correctly reproduced by our effective coarse-grained theory [cf. Eq. (9)]. Therefore, in the language of Tomonaga-Luttinger liquid physics, our treatment amounts to retaining only forward scattering processes.

Our results point towards a rich behavior of the 1D plasmon mode in the wire, determined by the diffusive modes in the electron gas. Independently of its dimensionality, in the static screening limit $|\omega_m| \ll Dq^2$, the wire has a plasmon excitation which follows approximately a linear dispersion relation. One could naively think that in that regime dissipative effects are negligible. However, the complete solutions of Eqs. (23) and (32) indicate that this is not the case. Indeed, we obtain sizable dissipative effects even in the limit $|\omega(q)| \ll Dq^2$, which manifests itself in the broadening of the 1D plasmon mode (cf. Figs. 6 and 8). Although technically challenging from the experimental point of view, this broadening could be seen in experiments of resonant inelastic Raman light-scattering[59] or in optical measurements of the dynamic conductivity or the reflection coefficient[60].

On the other hand, our results reveal that the dynamical screening regime $Dq^2 \ll |\omega_m|$ should be the most relevant for experimental realizations (cf. Figs. 2 and 4). This is more or less evident from the fact that the plasma mode essentially follows a linear dispersion in the limit $q \to 0$, while the boundary between the dynamical and the static screening regimes (determined by the diffusive modes in the electron gas) is $\sim Dq^2$. More importantly, in the limit $Dq^2 \ll |\omega_m|$ the dimensionality of the electron gas is of central importance to determine the low-energy properties of the wire. If the screening is provided by a 1DEG, its dissipative processes are more efficiently transferred to the superconducting wire in the limit $q \to 0$. As a consequence, the plasma mode becomes an ill-defined excitation and the superconductor shows a finite dc-conductivity in the limit $\omega = 0$ (cf. Figs. 6 and 7). This effect could be seen, e.g., in dc-transport experiments on capacitively coupled superconducting/normal wires systems (cf. Fig. 7).

When the screening is provided by a 2DEG, acoustic plasma modes with a vanishing width are recovered in the limit $q \to 0$, which allows to neglect the dissipative effects due to the Coulomb interaction with the metal (cf. Figs. 5 and 8). The reason for this lies in the existence of the additional degree of freedom in the electron gas (perpendicular momentum k), which produces (upon integration) a weakening of dissipation effects. At this point it is tempting to speculate that a semi-infinite 3D metal, or a superconducting wire embedded in a 3D normal matrix, would provide an additional degree of freedom (momentum k' perpendicular to the wire and to k), and would weaken further the impact of dissipation in the metal.

These remarks are relevant to works suggesting the possibility to stabilize the superconductivity in 1D systems by coupling them to a bath of normal quasiparticles[13,24,25]. In these works, the basic underlying physical idea is that the normal bath provides a source of friction for the phase field $\theta(\mathbf{x})$ which tends to quench its fluctuations and therefore, to favor super-

conductivity (very much like in the case of a resistively shunted Josephson junction[2,3,61]). However, little attention has been given up to now to the simultaneous dissipative effects induced by the Coulomb interaction with the electrons in the metal, which produce friction in the dual field $\rho_s(\mathbf{x})$, and therefore tends to increase phase fluctuations, thus deteriorating superconducting properties. In that sense, our results show that the best condition would be to screen the Coulomb interaction with a clean (i.e., large diffusion constant D) metallic film (rather than a wire). This result lends credence to the analysis made in Ref. 25, where it was assumed that the Coulomb interactions only renormalize the bare Luttinger parameters of a superconducting wire in contact with a 2D normal diffusive metal system.

Many other issues remain to be addressed to get an accurate physical description of a superconducting wire coupled to a dissipative electron gas, such as the aforementioned effect of topological excitations[12], localization effects in the gates as a consequence of disorder, simultaneous effect of Coulomb interactions and Andreev tunneling, etc. We expect that our results inspire other works along these lines.

Acknowledgments

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Appendix A: Derivation of the effective action

Although the derivation of the low-energy action for a superconductor has been studied by several authors[12,20,28,29], here we follow more closely the derivation of De Palo *et al.*[20]. Our starting point is the decoupling of the interaction terms appearing in H_{s}^0 and H_{int} [Eqs. (3) and (7) respectively] by the means of Hubbard-Stratonovich transformations (HSTs)

$$e^{-\int d\tau\ H_{\mathrm{int}}(\tau)} \propto \int \prod_{\nu} \mathcal{D}\left[\tilde{\rho}_\nu\right]\ e^{-\frac{1}{2}\sum_{\nu=\pm}\int d^4x_\mu d^4x'_\mu\ \left[v_\nu\left(x_\mu - x'_\mu\right)\right]^{-1}\tilde{\rho}_\nu(x_\mu)\tilde{\rho}_\nu\left(x'_\mu\right)+i\int d^4x_\mu\ \hat{\rho}_\nu(x_\mu)\tilde{\rho}_\nu(x_\mu)}, \tag{A1}$$

$$e^{|U|\int d\mathbf{r}d\tau\ \psi^*_\uparrow\psi^*_\downarrow\psi_\downarrow\psi_\uparrow} \propto \int \mathcal{D}\left[\Delta^*,\Delta\right]\ e^{-\int d^4x_\mu\ \frac{|\Delta(x_\mu)|^2}{|U|}+\int d^4x_\mu\ \Delta^*(x_\mu)\psi_\downarrow(x_\mu)\psi_\uparrow(x_\mu)+\psi^*_\uparrow(x_\mu)\psi^*_\downarrow(x_\mu)\Delta(x_\mu)}, \tag{A2}$$

where we have introduced the compact notation $x_\mu = (\mathbf{r},\tau)$ and the bosonic fields $\tilde{\rho}_\nu(x_\mu), \Delta^*(x_\mu), \Delta(x_\mu)$. The quantity $\left[v_\nu\left(x_\mu - x'_\mu\right)\right]^{-1}$ is a compact notation for the Fourier transform

$$\left[v_\nu\left(x_\mu - x'_\mu\right)\right]^{-1} = \frac{1}{\beta\Omega}\sum_{k^\mu}\frac{e^{ik^\mu\left(x_\mu - x'_\mu\right)}}{v_\nu\left(k^\mu\right)}, \tag{A3}$$

where $v_\nu(k^\mu) = \int d^4x_\mu e^{-ik^\mu\left(x_\mu - x'_\mu\right)}v_\nu\left(x_\mu - x'_\mu\right)$, with $v_\nu\left(x_\mu - x'_\mu\right)$ the potential $v_\nu\left(x_\mu - x'_\mu\right) \equiv v_\nu(\mathbf{r}-\mathbf{r}')\,\delta(\tau-\tau')$. Note that the mode $k^\mu = 0$, for which the above HST is formally ill-defined, can be safely ignored by considering the interaction with the positive ionic background in the system (not explicitly written here).

Our next step is to decouple the quadratic term $\tilde{\rho}_\nu(x_\mu)\,\tilde{\rho}_\nu\left(x'_\mu\right)$ in Eq. (A1) by the means of an extra HST. According to Ref. 20, this has the advantage of introducing the *physical densities* (symmetric and antisymmetric) of the problem (cf. Eq. 6). Then,

$$e^{-\frac{1}{2}\int d^4x_\mu d^4x'_\mu\ \left[v_\nu\left(x_\mu - x'_\mu\right)\right]^{-1}\tilde{\rho}_\nu(x_\mu)\tilde{\rho}_\nu\left(x'_\mu\right)} \propto \int \mathcal{D}\left[\rho_\nu\right]\ e^{-\frac{1}{2}\int d^4x_\mu d^4x'_\mu\ \rho_\nu(x_\mu)v_\nu\left(x_\mu - x'_\mu\right)\rho_\nu\left(x'_\mu\right)-i\int d^4x_\mu\ \tilde{\rho}_\nu(x_\mu)\rho_\nu(x_\mu)}. \tag{A4}$$

Note that the formal integration of the field $\tilde{\rho}_\nu$ gives the functional-delta function[30] $\delta\left[\hat{\rho}_\nu - \rho_\nu\right]$. This fact allows to interpret the HS fields ρ_ν as the physical electronic densities[20].

It is convenient to write the action of the system after these manipulations

$$S = \int d^4x_\mu \sum_\sigma \left\{ \psi^*_\sigma \left(\partial_\tau - \mu\right) \psi_\sigma + \frac{1}{2m} \left[\nabla \psi^*_\sigma\right] \left[\nabla \psi_\sigma\right] \right\} +$$
$$+ \int d^4x_\mu \left\{ \frac{\left|\Delta\left(x_\mu\right)\right|^2}{|U|} - \Delta^*\left(x_\mu\right) \psi_\downarrow \psi_\uparrow - \psi^*_\uparrow \psi^*_\downarrow \Delta\left(x_\mu\right) \right\} +$$
$$+ \int dx_\mu \sum_\sigma \left\{ \eta^*_\sigma \left(\partial_\tau - \mu + V_{\rm i}\right) \eta_\sigma + \frac{1}{2m} \left[\nabla \eta^*_\sigma\right] \left[\nabla \eta_\sigma\right] \right\} +$$
$$+ \frac{1}{2} \sum_{\nu=\pm} \int d^4x_\mu d^4x'_\mu \, \rho_\nu\left(x_\mu\right) v_\nu\left(x_\mu - x'_\mu\right) \rho_\nu\left(x'_\mu\right) +$$
$$+ i \sum_{\nu=\pm} \int d^4x_\mu \, \tilde{\rho}_\nu\left(x_\mu\right) \left[\rho_\nu\left(x_\mu\right) - \hat{\rho}_\nu\left(x_\mu\right)\right], \tag{A5}$$

where for simplicity we have dropped the arguments in the fermionic fields ψ_σ and η_σ and in the disorder potential $V_{\rm i} = V_{\rm i}\left(\mathbf{r}\right)$.

The next step is to perform the saddle-point approximation with respect to the bosonic fields $\Delta\left(x_\mu\right), \Delta^*\left(x_\mu\right), \tilde{\rho}_\nu\left(x_\mu\right), \rho_\nu\left(x_\mu\right)$, which gives the equations

$$\frac{\delta S}{\delta \Delta^*\left(x_\mu\right)} = 0 = \frac{\Delta\left(x_\mu\right)}{|U|} - \psi_\downarrow\left(x_\mu\right) \psi_\uparrow\left(x_\mu\right), \tag{A6}$$
$$\frac{\delta S}{\delta \Delta\left(x_\mu\right)} = 0 = \frac{\Delta^*\left(x_\mu\right)}{|U|} - \psi^*_\uparrow\left(x_\mu\right) \psi^*_\downarrow\left(x_\mu\right), \tag{A7}$$
$$\frac{\delta S}{\delta \tilde{\rho}_\nu\left(x_\mu\right)} = 0 = \rho_\nu\left(x_\mu\right) - \hat{\rho}_\nu\left(x_\mu\right), \tag{A8}$$
$$\frac{\delta S}{\delta \rho_\nu\left(x_\mu\right)} = 0 = \int dx'_\mu \, v_\nu\left(x_\mu - x'_\mu\right) \rho_\nu\left(x'_\mu\right) + i \tilde{\rho}_\nu\left(x_\mu\right). \tag{A9}$$

The first two equations reproduce the well-known BCS gap-equation[32], while the other two give the relationship between $\rho_\nu, \tilde{\rho}_\nu$ and the electronic density. These equations provide the starting point for a controlled expansion in terms of Gaussian fluctuations of the bosonic fields around the uniform solutions $\Delta^{(0)}$, $\rho^{(0)}_\nu$ and $\tilde{\rho}^{(0)}_\nu$. In what follows, we assume that the values of $\Delta^{(0)}$, $\rho^{(0)}_\nu$ and $\tilde{\rho}^{(0)}_\nu$ are known. When these solutions are inserted back into the action Eq. (A5), we notice that the quantity $\tilde{\rho}^{(0)}_\nu = i \rho^{(0)}_\nu \int d^4x_\mu \, v_\nu\left(x_\mu\right)$ can be absorbed in a renormalization of the chemical potential μ_ν due to the effect of Coulomb interactions, while the divergent quantity $\frac{1}{2} \sum_{\nu=\pm} \left(\rho^{(0)}_\nu\right)^2 \int d^4x_\mu d^4x'_\mu \, v_\nu\left(x_\mu - x'_\mu\right)$ exactly cancels the contribution coming from the positive ionic background (which we have not written explicitly here), by imposing the overall electroneutrality of the system, and consequently we will drop it in the following. We also drop the constant term $\beta \Omega \frac{\Delta_0^2}{|U|}$, where Ω is the volume of the superconducting system.

At sufficiently low energies, amplitude fluctuations of the order parameter can be neglected, and we can write $\Delta\left(x_\mu\right) = \Delta_0 e^{i\theta\left(x_\mu\right)}$, with a real constant $\Delta_0 = \left|\Delta^{(0)}\right|$. We can absorbe the phase field by the means of a transformation of the fermion field

$$\psi_\sigma\left(x_\mu\right) \to \psi'_\sigma\left(x_\mu\right) = \psi_\sigma\left(x_\mu\right) e^{i\theta\left(x_\mu\right)/2}.$$

The expression of the effective action is considerably simplified introducing the Nambu notation

$$\mathbf{\Psi}\left(x_\mu\right) \equiv \begin{pmatrix} \psi_\uparrow\left(x_\mu\right) \\ \psi^*_\downarrow\left(x_\mu\right) \end{pmatrix}, \quad \boldsymbol{\eta}\left(x_\mu\right) \equiv \begin{pmatrix} \eta_\uparrow\left(x_\mu\right) \\ \eta^*_\downarrow\left(x_\mu\right) \end{pmatrix},$$

which allows to write the action as

$$S \simeq \int d^4x_\mu \left\{ \mathbf{\Psi}^\dagger \left[\boldsymbol{\mathcal{A}}_{0,s} - \mathbf{\Sigma}_s\right] \mathbf{\Psi} + \boldsymbol{\eta}^\dagger \left[\boldsymbol{\mathcal{A}}_{0,n} - \mathbf{\Sigma}_n\right] \boldsymbol{\eta} \right\} +$$
$$+ \frac{1}{2} \sum_\nu \int d^4x_\mu d^4x'_\mu \, \delta\rho_\nu\left(x_\mu\right) v_\nu\left(x_\mu - x'_\mu\right) \delta\rho_\nu\left(x'_\mu\right) +$$
$$+ i \sum_{\nu=\pm} \int d^4x_\mu \, \delta\tilde{\rho}_\nu\left(x_\mu\right) \left[\rho^{(0)}_\nu + \delta\rho_\nu\left(x_\mu\right)\right].$$

where

$$\delta\tilde{\rho}_\nu\left(x_\mu\right) \equiv \tilde{\rho}_\nu\left(x_\mu\right) - \tilde{\rho}^{(0)}_\nu, \tag{A10}$$
$$\delta\rho_\nu\left(x_\mu\right) \equiv \rho_\nu\left(x_\mu\right) - \rho^{(0)}_\nu, \tag{A11}$$

are the fluctuations of the density around the saddle-point solutions, and

$$\boldsymbol{\mathcal{A}}_{0,s} \equiv \left\{\partial_\tau\right\} \hat{\boldsymbol{\tau}}_0 - \left\{ \frac{\nabla^2}{2m} + \mu_s \right\} \hat{\boldsymbol{\tau}}_3 - \Delta_0 \hat{\boldsymbol{\tau}}_1,$$
$$\mathbf{\Sigma}_s \equiv - \left\{ \frac{i\left(\partial_\tau \theta\right)}{2} + \frac{\left(\nabla\theta\right)^2}{8m} - i \sum_\nu \delta\tilde{\rho}_\nu \right\} \hat{\boldsymbol{\tau}}_3 + \frac{i\left(\nabla\theta\right)\nabla}{2m} \hat{\boldsymbol{\tau}}_0,$$
$$\boldsymbol{\mathcal{A}}_{0,n} \equiv \left\{\partial_\tau\right\} \hat{\boldsymbol{\tau}}_0 - \left\{ \frac{\nabla^2}{2m} + \mu_n \right\} \hat{\boldsymbol{\tau}}_3,$$
$$\mathbf{\Sigma}_n \equiv i \sum_\nu \left(\nu\right) \delta\tilde{\rho}_\nu \hat{\boldsymbol{\tau}}_3,$$

where $\hat{\boldsymbol{\tau}}_i$ are the Pauli matrices and where we have used the fact that $\frac{i\nabla\theta\left[\overrightarrow{\nabla} - \overleftarrow{\nabla}\right]}{2m} = \frac{i\nabla\theta\nabla}{m}$ in a translationally invariant system.

The next step consists in using the expansion formula

$$\mathrm{Tr}\, \ln\left[\boldsymbol{\mathcal{A}}_0 - \mathbf{\Sigma}\right] = \mathrm{Tr}\, \ln \boldsymbol{\mathcal{A}}_0 - \sum_{n=1}^{\infty} \frac{\left(-1\right)^n}{n} \mathrm{Tr}\, \left[\mathbf{G}_0 \mathbf{\Sigma}\right]^n,$$

where $\mathbf{G}_0 \equiv - \left[\boldsymbol{\mathcal{A}}_0\right]^{-1}$. Truncating the series at second order (i.e., Gaussian fluctuations), we obtain

$$S \simeq -\mathrm{Tr}\, \left[\mathbf{G}_{0,s} \mathbf{\Sigma}_s\right] + \frac{1}{2} \mathrm{Tr}\, \left[\mathbf{G}_{0,s} \mathbf{\Sigma}_s\right]^2 +$$
$$- \mathrm{Tr}\, \left[\mathbf{G}_{0,n} \mathbf{\Sigma}_n\right] + \frac{1}{2} \mathrm{Tr}\, \left[\mathbf{G}_{0,n} \mathbf{\Sigma}_n\right]^2 +$$
$$+ \frac{1}{2} \sum_\nu \int d^4x_\mu d^4x'_\mu \, \delta\rho_\nu\left(x_\mu\right) v_\nu\left(x_\mu - x'_\mu\right) \delta\rho_\nu\left(x'_\mu\right) +$$
$$+ i \sum_{\nu=\pm} \int d^4x_\mu \, \delta\tilde{\rho}_\nu\left(x_\mu\right) \left[\rho^{(0)}_\nu + \delta\rho_\nu\left(x_\mu\right)\right],$$

where the propagators in Nambu space $\mathbf{G}_{0,s}$ and $\mathbf{G}_{0,n}$ write

$$\mathbf{G}_{0,s} = \begin{pmatrix} g_{0,s}(x_\mu) & f_{0,s}(x_\mu) \\ \bar{f}_{0,s}(x_\mu) & \bar{g}_{0,s}(x_\mu) \end{pmatrix},$$
$$\mathbf{G}_{0,n} = \begin{pmatrix} g_{0,n}(x_\mu) & 0 \\ 0 & \bar{g}_{0,n}(x_\mu) \end{pmatrix},$$

and where $g_{0,s}(x_\mu) \equiv -\left\langle T_\tau \psi_\uparrow(x_\mu)\psi^*_\uparrow(0)\right\rangle$ and $\bar{g}_{0,s}(x_\mu) \equiv \left\langle T_\tau \psi^*_\downarrow(x_\mu)\psi_\downarrow(0)\right\rangle$ denote respectively the particle and hole propagators in the superconductor, while $f_{0,s}(x_\mu) \equiv \langle T_\tau \psi_\downarrow(x_\mu)\psi_\uparrow(0)\rangle$, $\bar{f}_{0,s}(x_\mu) \equiv \left\langle T_\tau \psi^*_\uparrow(x_\mu)\psi^*_\downarrow(0)\right\rangle$ are the anomalous ones[62]. Similarly $g_{0,n}(x_\mu) \equiv -\left\langle T_\tau \eta_\uparrow(x_\mu)\eta^*_\uparrow(0)\right\rangle$ and $\bar{g}_{0,n}(x_\mu) \equiv \left\langle T_\tau \eta^*_\downarrow(x_\mu)\eta_\downarrow(0)\right\rangle$ are the particle and hole propagators in the normal metal, respectively.

The evaluation of the traces yields

$$\mathrm{Tr}\left[\mathbf{G}_{0,s}\mathbf{\Sigma}_s\right] = -\rho_s^{(0)}\int d^4x_\mu \left[\frac{i}{2}\partial_\tau\theta + \frac{(\nabla\theta)^2}{8m} - i\sum_\nu \delta\tilde{\rho}_\nu\right]_{x_\mu},$$
$$\mathrm{Tr}\left[\mathbf{G}_{0,s}\mathbf{\Sigma}_s\right]^2 = \int d^4x_\mu d^4x'_\mu \left\{\chi_{0,s}\left(x_\mu - x'_\mu\right)\times\right.$$
$$\times\left[\frac{1}{2}\partial_\tau\theta - \sum_\nu \delta\tilde{\rho}_\nu\right]_{x_\mu}\left[\frac{1}{2}\partial_\tau\theta - \sum_{\nu'} \delta\tilde{\rho}_{\nu'}\right]_{x'_\mu} +$$
$$\left. - \mathcal{D}'\left(x_\mu - x'_\mu\right)\left[\nabla\theta\right]_{x_\mu}\left[\nabla\theta\right]_{x'_\mu}\right\},$$
$$\mathrm{Tr}\left[\mathbf{G}_{0,n}\mathbf{\Sigma}_n\right] = i\rho_n^{(0)}\int d^4x_\mu \left[\sum_\nu (\nu)\,\delta\tilde{\rho}_\nu\right]_{x_\mu},$$
$$\mathrm{Tr}\left[\mathbf{G}_{0,n}\mathbf{\Sigma}_n\right]^2 = \int d^4x_\mu d^4x'_\mu\, \chi_{0,n}\left(x_\mu - x'_\mu\right)\times$$
$$\times\left[\sum_\nu (\nu)\,\delta\tilde{\rho}_\nu\right]_{x_\mu}\left[\sum_{\nu'} (\nu')\,\delta\tilde{\rho}_{\nu'}\right]_{x'_\mu},$$

where for simplicity we have used the campact notation $[\mathcal{A}]_{x_\mu} \equiv \mathcal{A}(x_\mu)$, and where we have defined

$$\mathcal{D}'\left(x_\mu - x'_\mu\right) \equiv \frac{1}{2m^2}\left[\nabla g_{0,s}\left(x'_\mu - x_\mu\right)\nabla g_{0,s}\left(x_\mu - x'_\mu\right) + \right.$$
$$\left. + \nabla f_{0,s}\left(x'_\mu - x_\mu\right)\nabla f_{0,s}\left(x_\mu - x'_\mu\right)\right],$$

and the density-density correlation functions

$$\chi_{0,s}\left(x_\mu - x'_\mu\right) \equiv -\left\langle T_\tau \delta\rho_s(x_\mu)\,\delta\rho_s\left(x'_\mu\right)\right\rangle,$$
$$= -2g_{0,s}\left(x_\mu - x'_\mu\right)g_{0,s}\left(x'_\mu - x_\mu\right) +$$
$$+ 2f_{0,s}\left(x_\mu - x'_\mu\right)f_{0,s}\left(x'_\mu - x_\mu\right), \quad \text{(A12)}$$
$$\chi_{0,n}\left(x_\mu - x'_\mu\right) \equiv -\left\langle T_\tau \delta\rho_n(x_\mu)\,\delta\rho_n\left(x'_\mu\right)\right\rangle,$$
$$= -2g_{0,n}\left(x_\mu - x'_\mu\right)g_{0,n}\left(x'_\mu - x_\mu\right). \quad \text{(A13)}$$

The final step is to integrate out the modes $\delta\tilde{\rho}_\nu(x_\mu)$. To that aim, we decouple the mixed term $\sim (\delta\tilde{\rho}_\nu)(\delta\tilde{\rho}_{\bar{\nu}})'$ appearing in $\mathrm{Tr}\left[\mathbf{G}_{0,s}\mathbf{\Sigma}_s\right]^2$ and $\mathrm{Tr}\left[\mathbf{G}_{0,n}\mathbf{\Sigma}_n\right]^2$ by returning to the original representation for the densities [cf. Eq. (6)]

$$\delta\tilde{\rho}_s = \frac{\delta\tilde{\rho}_+ + \delta\tilde{\rho}_-}{2},$$
$$\delta\tilde{\rho}_n = \frac{\delta\tilde{\rho}_+ - \delta\tilde{\rho}_-}{2},$$

and integrate out the fields $\delta\tilde{\rho}_n(x_\mu)$ and $\delta\tilde{\rho}_s(x_\mu)$ instead. From here we see that the term $\sim (\partial_\tau\theta)(\partial_\tau\theta)'$ cancels, as in Ref. 20. We finally obtain

$$S = \frac{i}{2}\int d^4x_\mu\, \partial_\tau\theta(x_\mu)\,\rho_s(x_\mu) +$$
$$+ \int d^4x_\mu d^4x'_\mu \left\{\frac{1}{2}\mathcal{D}\left(x_\mu - x'_\mu\right)\nabla\theta(x_\mu)\nabla\theta\left(x'_\mu\right) + \right.$$
$$\left. + \boldsymbol{\delta\rho}^\dagger(x_\mu)\,\boldsymbol{V}\left(x_\mu - x'_\mu\right)\boldsymbol{\delta\rho}\left(x'_\mu\right)\right\} \quad \text{(A14)}$$

where we have defined

$$\mathcal{D}(x_\mu) \equiv \frac{\rho_s^{(0)}}{4m}\delta(x_\mu) - \mathcal{D}'(x_\mu), \quad \text{(A15)}$$
$$\boldsymbol{\delta\rho}(x_\mu) \equiv \begin{pmatrix} \delta\rho_s(x_\mu) \\ \delta\rho_n(x_\mu) \end{pmatrix},$$
$$\boldsymbol{V}(x_\mu) \equiv \begin{pmatrix} \left[\chi_{0,s}(x_\mu)\right]^{-1} & 0 \\ 0 & \left[\chi_{0,n}(x_\mu)\right]^{-1} \end{pmatrix} +$$
$$+ \delta(\tau)\begin{pmatrix} v(x_\mu, 0) & v(x_\mu, d) \\ v(x_\mu, d) & v(x_\mu, 0) \end{pmatrix},$$

where again we used a compact notation of Eq. (A3).

Appendix B: Density susceptibility and superconducting stiffness in the limit $q \to 0$

From Eq. (A12), the Fourier transforms reads

$$\chi_{0,s}(q^\mu) = \frac{2}{\beta\Omega}\sum_{k^\mu}\left[-g_{0,s}(k^\mu)\,g_{0,s}(k^\mu - q^\mu) + \right.$$
$$\left. + f_{0,s}(k^\mu)\,f_{0,s}(k - q^\mu)\right].$$

In the limit $q^\mu \to 0$, we obtain

$$\lim_{q^\mu\to 0}\chi_{0,s}(q^\mu) \to -\frac{2}{\Omega}\sum_{\mathbf{k}}\frac{1}{\beta}\sum_n \frac{(i\nu_n)^2 + \xi^2_{\mathbf{k}} - \Delta_0^2}{\left[(i\nu_n)^2 - E^2_{\mathbf{k}}\right]^2}$$
$$= -\frac{2}{\Omega}\left[\sum_k \frac{n_F(E_{\mathbf{k}})}{2E_{\mathbf{k}}} - \frac{n_F(-E_{\mathbf{k}})}{2E_{\mathbf{k}}}\right],$$

with $\xi_{\mathbf{k}} \equiv \frac{k^2}{2m} - \mu_s$, and $E_{\mathbf{k}} \equiv \sqrt{\xi_{\mathbf{k}}^2 + \Delta_0^2}$. At $T = 0$

$$\begin{aligned}\lim_{q^\mu \to 0} \chi_{0,s}(q^\mu) &= \frac{2}{\Omega} \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}}}. \\ &= \mathcal{N}_s^{(0)} \gamma, \end{aligned} \tag{B1}$$

where $\gamma \equiv \int_{-\omega_D}^{\omega_D} d\xi \frac{1}{\sqrt{\xi^2+\Delta_0^2}} = \ln\left[\frac{\omega_D + \sqrt{\omega_D^2+\Delta_0^2}}{-\omega_D+\sqrt{\omega_D^2+\Delta_0^2}}\right] \approx 2\ln\left[\frac{2\omega_D}{\Delta_0}\right]$, and where ω_D is a high-energy cutoff.

Similarly, the Fourier transform of the superconducting stiffness reads

$$\mathcal{D}(q^\mu) \equiv \frac{\rho_s^{(0)}}{4m} - \mathcal{D}'(q^\mu),$$

with

$$\begin{aligned}\mathcal{D}'(q^\mu) &\equiv \frac{1}{\Omega} \sum_{k^\mu} \frac{-\mathbf{k}.(\mathbf{k}-\mathbf{q})}{2m^2} \times \\ &\times \left[f(k^\mu) f(k^\mu - q^\mu) + g(k^\mu) g(k^\mu - q^\mu)\right] \\ &= \frac{1}{V} \sum_{\mathbf{k}} \frac{-\mathbf{k}.(\mathbf{k}-\mathbf{q})}{2m^2} \times \\ &\times \frac{1}{\beta} \sum_n \frac{\Delta_0^2 + (i\nu_n + \xi_{\mathbf{k}})(i\nu_n - i\omega_m + \xi_{\mathbf{k}-\mathbf{q}})}{\left[(i\nu_n)^2 - E_{\mathbf{k}}^2\right]\left[(i\nu_n - i\omega_m)^2 - E_{\mathbf{k}-\mathbf{q}}^2\right]},\end{aligned}$$

Evaluating the Matsubara sum over the fermionic frequencies $i\nu_n$, we obtain the result in the limit $q^\mu \to 0$

$$\begin{aligned}\lim_{q^\mu \to 0} \mathcal{D}'(q^\mu) &\approx -\frac{1}{V} \sum_{\mathbf{k}} \frac{k^2}{2m^2} \frac{n_F(E_{\mathbf{k}}) - n_F(E_{\mathbf{k}-\mathbf{q}})}{E_{\mathbf{k}} - E_{\mathbf{k}-\mathbf{q}}} \\ &\approx -\frac{1}{V} \sum_{\mathbf{k}} \frac{k^2}{2m^2} \frac{\partial n_F(E_{\mathbf{k}})}{\partial E_{\mathbf{k}}},\end{aligned}$$

which vanishes in the limit $T \to 0$, and we recover the well-know result[32,62]

$$\lim_{q^\mu \to 0} \mathcal{D}(q^\mu) \equiv \mathcal{D}_0 = \frac{\rho_s^{(0)}}{4m}.$$

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